MATHEMATICS

LOCAL AND GLOBAL HYPERCENTRALITY AND SUPERSOLUBILITY II

BY

REINHOLD BAER

Dedicated to Hans Freudenthal on the occasion of his 60th birthday

(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of September 25, 1965)

G. The weak normalizer condition which we are going to discuss next plays the same rôle for supersolubility as the normalizer condition for hypercentrality. We say that the group G satisfies the *weak normalizer condition,* if it meets the following requirement:

(G.0) *If X is a subgroup of G with $\mathfrak{n}X = X \subset G$, then there exists an element t in G, but not in X, with*

$$\{t^X\} \subseteq X_{\{X,t\}} \; \{t\}.$$

Here X_Y for X a subgroup of Y signifies *the core of X in Y.* This is at the same time the product of all the normal subgroups of Y which are part of X and the intersection of all the subgroups conjugate to X in Y. It is in short the most comprehensive normal subgroup of Y which is part of X.

It is clear that the normalizer condition implies the weak normalizer condition; and that the converse is false, we shall see below.

It is not difficult so see that the weak normalizer condition is inherited by epimorphic images. But whether it is inherited by subgroups we have not been able to decide.

Lemma G.1: *The weak normalizer condition is satisfied by every supersoluble group.*

Proof: Suppose that $U \subset G$. Then $G/U_G \neq 1$ so that G/U_G possesses a cyclic normal subgroup $Z/U_G \neq 1$. There exists an element t with $Z/U_G = \{U_G t\}$ and this is equivalent with $Z = U_G\{t\}$. If t were an element of U, then the normal subgroup Z of G were part of U, implying the contradiction

$$U_G \subset Z \subseteq U_G.$$

Hence t does not belong to U. Since $U_{\{U,t\}}$ is the intersection of all the subgroups conjugate to U in $\{U, t\}$ whereas U_G is the intersection of all the subgroups conjugate to U in G, we have $U_G \subseteq U_{\{U,t\}}$. Since Z is a normal subgroup of G and t is in Z, we have

$$\{t^U\} \subseteq \{t^G\} \subseteq Z = U_G\{t\} \subseteq U_{\{U,t\}}\{t\};$$

and this proves the weak normalizer condition [and more].

Excursus on finite supersoluble groups: To show the force of the weak normalizer condition and to justify its name we shall prove the following analogue of Wielandt's characterization of finite nilpotent groups.

The following properties of the finite group G are equivalent:

(i) *G is supersoluble.*

(ii) *The weak normalizer condition is satisfied by G.*

(iii) *If the maximal subgroup X of G is not a normal subgroup of G, then there exists an element g in G with*

$$G = \{X, g\} \text{ and } \{g^G\} \subseteq X_G\{g\}.$$

Proof: It is a consequence of Lemma G.1 that (i) implies (ii). – If X is a maximal subgroup of G, then either X is a normal subgroup of G or else $X = \mathfrak{n}X$; and furthermore $G = \{X, g\}$ for every element g, not in X. Now it is clear that (iii) is a consequence of (ii).

Assume finally the validity of (iii) and consider a maximal subgroup X of G. If firstly X is a normal subgroup of G, then G/X is free of proper subgroups so that $[G\colon X]$ is a prime. If secondly X is not a normal subgroup of G, then $X = \mathfrak{n}X$ and there exists by (iii) an element g in G with

$$G = \{X, g\} \text{ and } \{g^G\} \subseteq X_G\{g\}.$$

Let $G^* = G/X_G$, $X^* = X/X_G$ and $g^* = X_G g$. Then X^* is a maximal subgroup of G^* and $\{g^*\} = \{g^{*G^*}\}$ is a cyclic normal subgroup of G^* with $G^* = X^*\{g^*\}$. Since every subgroup of a cyclic group is a characteristic subgroup, every subgroup of the cyclic normal subgroup $\{g^*\}$ is a normal subgroup of G^*. From $X^*_{G^*} = 1$ we deduce now $X^* \cap \{g^*\} = 1$. If S is a subgroup with $1 \subset S \subset \{g^*\}$, then S is a normal subgroup of G^* with $X^* \subset X^*S \subset G^*$, contradicting the maximality of X^*. Hence $\{g^*\}$ is free of proper subgroups and is consequently of order a prime, implying that $[G\colon X] = [G^*\colon X^*]$ is a prime. Thus we have shown that every maximal subgroup of G has index a prime. Application of Huppert's Theorem – see Hall [p. 162, Theorem 10.5.8] – shows the supersolubility of G.

Theorem G.2: *The following properties of the group G are equivalent.*

(i) *G is finitely generated and supersoluble.*

(ii) *G is noetherian and the weak normalizer condition is satisfied by G.*

(iii) *G is finitely generated; $G^{(i)} = 1$ for almost all i; and the weak normalizer condition is satisfied by every factor of G.*

Remark: It is still an open question whether it suffices to require in (ii) that G be finitely generated and whether it suffices to require in (iii) that the weak normalizer condition be satisfied by G. — Instead of requiring in (ii) that G be noetherian it suffices to assume that the maximum condition is satisfied by the soluble subgroups of G.

We have constructed in Baer [9; p. 26/27, § 4] an example of a group G with the following properties:

G is generated by two elements and $G'' = 1$.

There exists a torsionfree abelian normal subgroup A of rank 1 with infinite cyclic G/A.

G is neither supersoluble not noetherian.

From the second of these properties one deduces that every finite factor of G is supersoluble. This shows the impossibility of weakening condition (iii) by requiring that the weak normalizer condition be satisfied by all finite factors [instead of requiring it of all factors]. A similar remark may be appended to some of our later results.

Proof: If G is a finitely generated supersoluble group, then we deduce from Lemma G.1 the validity of the weak normalizer condition; and it is a consequence of Baer [3; p. 26, Theorem 1] that G is noetherian.

Next we assume the validity of (ii) and we are going to prove the supersolubility of G. Term a group X soluble, if almost all its derivatives $X^{(i)} = 1$. Since G is noetherian, there exists a maximal soluble subgroup M of G. If M is normalized by the element g, then M is a normal subgroup of $\{M, g\}$ with cyclic $\{M, g\}/M$. Hence $\{M, g\}$ is likewise soluble so that $M = \{M, g\}$ by the maximality of M. Thus g belongs to M and we have shown $M = \mathfrak{n}M$. Assume now by way of contradiction that $M \subset G$. Then there exists by (G.0) an element t in G which does not belong to M with

$$\{t^M\} \subseteq M_{\{M,t\}}\{t\}.$$

The solubility of M implies first that of $M_{\{M,t\}}$, then that of $M_{\{M,t\}}\{t\}$, then that of the normal subgroup $\{t^M\}$ of $\{M, t\}$. But

$$\{M, t\}/\{t^M\} = \{t^M\}\, M/\{t^M\} \cong M/[M \cap \{t^M\}]$$

is likewise soluble so that $\{M, t\}$ is soluble as an extension of a noetherian soluble group by a soluble group. Hence $M = \{M, t\}$ by the maximality of M, a contradiction proving $M = G$. Thus we have shown that

(1) G is soluble.

Consider a maximal subgroup S of G. If S is a normal subgroup of G, then G/S is free of proper subgroups and hence cyclic of order a prime. Consequently $[G\colon S]$ is a prime. — If S is not a normal subgroup of G, then $S = \mathfrak{n}S$ is a consequence of the maximality of S. Because of the weak normalizer condition there exists an element s which does not belong to

S such that $\{s^S\} \subseteq S_{\{S,s\}}\{s\}$. Because of the maximality of S we have $\{S, s\} = G$ so that

$$\{s^G\} = \{s^S\} \subseteq S_G\{s\}.$$

If σ is the canonical epimorphism of G upon $H = G/S_G$, then $S^\sigma = S/S_G$ and $\{s^\sigma\} = \{s^G\}^\sigma$ is a normal subgroup of H. We note furthermore that $T = S^\sigma$ is a maximal subgroup of H with $T_H = 1$. Since every subgroup of a cyclic subgroup is a characteristic subgroup, every subgroup of the cyclic normal subgroup $\{s^\sigma\}$ of H is a normal subgroup of H so that $T \cap \{s^\sigma\} \subseteq T_H = 1$. Furthermore

$$T\{s^\sigma\} = \{S, s\}^\sigma = G^\sigma = H.$$

If $X \neq 1$ is a subgroup of $\{s^\sigma\}$, then X is a normal subgroup of H with $T \cap X = T \cap \{s^\sigma\} \cap X = 1$ and $TX = H$ because of the maximality of T. Application of Dedekind's modular law shows $X = \{s^\sigma\}$. Hence $\{s^\sigma\}$ is free of proper subgroups. Since $\{s^\sigma\} \neq 1$, it is of order a prime p. Consequently

$$p = o(\{s^\sigma\}) = [H : T] = [G : S]$$

and we have shown that

(2) every maximal subgroup of G has index a prime in G.

Elsewhere – Baer [3; p. 11, Einleitung] – we have shown:

The group G is noetherian and supersoluble if, and only if,

(A) G is finitely generated,

(B) the maximal subgroups of G have index a prime and

(C) every infinite epimorphic image of G possesses a finitely generated abelian normal subgroup, not 1.

Our group G meets requirement (A), since G is supposed to be noetherian. Requirement (B) is a consequence of (2) and (C) is true, since G is, by hypothesis, noetherian and, by (1), soluble. Thus our group G is supersoluble.

If the equivalent conditions (i) and (ii) are satisfied by G, then the noetherian and supersoluble group G is soluble – see (1) – and every factor of G is likewise supersoluble and satisfies by Lemma G.1 the weak normalizer condition. Hence (iii) is a consequence of (i), (ii).

We precede the proof of the fact that (iii) implies (i), (ii) by the proofs of some independent auxiliary propositions.

(3) If the weak normalizer condition is satisfied by every factor of the group X, then every finite factor of X is supersoluble.

This is a consequence of the equivalence of (i) and (ii) already proven.

(4) If the weak normalizer condition is satisfied by every factor of the group X and if X is an extension of an abelian group by a cyclic group, then X is supersoluble.

Proof: Assume that the epimorphic image H of X is not cyclic. Then there exists an abelian normal subgroup A of H with cyclical H/A and the weak normalizer condition is inherited by H. Hence there exists an element e with $H=A\{e\}$.

If $\mathfrak{z}H\neq 1$, then there exists a cyclic normal subgroup, not 1, of H. Hence we assume that $\mathfrak{z}H=1$. Since A is abelian, we have $A \cap \mathfrak{c}e \subseteq \mathfrak{z}H=1$ [because of $H=A\{e\}$]. It follows that $A \cap \mathfrak{c}e=1$.

Consider an element a in $A \cap \mathfrak{n}\{e\}$. Since A is a normal subgroup of H, the commutator $a \circ e$ belongs to

$$A \cap \{e\} \subseteq A \cap \mathfrak{c}e=1$$

so that $a \circ e=1$. Hence a belongs to $A \cap \mathfrak{c}e=1$; and we have shown that $A \cap \mathfrak{n}\{e\}=1$. From $\{e\} \subseteq \mathfrak{n}\{e\} \subseteq H=A\{e\}$ and Dedekind's modular law we deduce that

$$\mathfrak{n}\{e\}=\{e\}[\mathfrak{n}\{e\} \cap A]=\{e\}.$$

Next we note that H is not cyclical and that therefore $\{e\} \subset H$. Application of the weak normalizer condition shows the existence of an element t in H, not in $\{e\}$, with

$$\{t^{\{e\}}\} \subseteq \{e\}_{\{e,t\}}\{t\}.$$

Since $\{e\}_{\{e,t\}}\{t\}$ is an extension of a cyclic group by a cyclic group, it is noetherian. Hence $\{t^{\{e\}}\}$ is likewise noetherian. Hence $\{t, e\}$ is noetherian as an extension of its noetherian normal subgroup $\{t^{\{e\}}\}$ by a cyclic group. Since the weak normalizer condition is satisfied by the subgroup $\{t, e\}$ of H, application of the equivalence of (i) and (ii) shows the supersolubility of $\{t, e\}$. From $\{e\} \subset \{t, e\} \subseteq H=A\{e\}$ and the normality of A we conclude that $A \cap \{t, e\}$ is a normal subgroup, not 1, of $\{t, e\}$. But then there exists a cyclic normal subgroup Z of $\{t, e\}$ with $1 \subset Z \subseteq \{t, e\} \cap A$; see Baer [3; p. 17, Lemma 2]. Since Z is centralized by the abelian group A and normalized by e, it is normalized by $A\{e\}=H$; and this proves the supersolubility of X.

(5) If the weak normalizer condition is satisfied by every factor of the finitely generated group X and if X is an extension of an abelian group by a noetherian supersoluble group, then X is noetherian and supersoluble.

Proof: Assume that $H\neq 1$ is an epimorphic image of X. Then there exists an abelian normal subgroup A of H with noetherian and supersoluble H/A. If $A=1$, then H certainly possesses a cyclic normal subgroup, not 1. Thus we assume $A\neq 1$. Since H/A is noetherian and supersoluble, there exist normal subgroups $A(i)$ of H with

$$A=A(0),\ A(i+1)/A(i) \text{ is cyclic, } A(n)=H.$$

We prove by complete induction with respect to i that A contains a finitely generated normal subgroup, not 1, of $A(i)$. This is certainly true for $i=0$, since $A(0)=A\neq 1$ is abelian. Assume now that $i<n$ and that Z is a finitely generated normal subgroup of $A(i)$ with $1\subset Z\subseteq A$. Since $A(i+1)/A(i)$ is cyclic, there exists an element e with $A(i+1)=A(i)\{e\}$. Next we note that

$$\{Z^{\{e\}}\}^{A(i)}=\{Z^{\{e\}A(i)}\}=\{Z^{A(i)\{e\}}\}=\{Z^{\{e\}}\},$$

since Z is normalized by $A(i)$ and $A(i)$ by $\{e\}$. Hence $N=\{Z^{\{e\}}\}$ is a normal subgroup of $A(i+1)$; and since $Z\subseteq A$, we have $N\subseteq A$. Consequently $\{Z, e\}$ is a finitely generated extension of its abelian normal subgroup N by a cyclic group. Application of (4) shows that $\{Z, e\}$ is supersoluble and hence, by Baer [3; p. 26, Theorem 1], noetherian. This implies in particular that N is noetherian; and thus we have shown the existence of a finitely generated normal subgroup N of $A(i+1)$ with $1\subset N\subseteq A$. This completes the inductive argument, proving that

(A) every epimorphic image, not 1, of X possesses a finitely generated abelian normal subgroup, not 1.

We deduce from (3) that

(B) every finite epimorphic image of X is supersoluble.

By hypothesis finally

(C) X is finitely generated.

But we have shown elsewhere—Baer [9; p. 20, Satz 3.1]—that a group, meeting requirements (A)–(C), is noetherian and supersoluble. This completes the proof of (5).

(6) If the weak normalizer condition is satisfied by every factor of the finitely generated group X, then $X/X^{(i)}$ is noetherian and supersoluble for every i.

Proof: This is certainly true for $i=0$. If we have already shown that $X/X^{(i)}$ is noetherian and supersoluble, then we note that $X/X^{(i+1)}$ is a finitely generated extension of the abelian group $X^{(i)}/X^{(i+1)}$ by the noetherian and supersoluble group $X/X^{(i)}$ and that the weak normalizer condition is satisfied by the factors of $X/X^{(i+1)}$. Application of (5) shows that $X/X^{(i+1)}$ is noetherian and supersoluble, completing the inductive proof of (6).

It is an immediate consequence of (6) that (iii) implies (i).

Corollary G.3: *The following properties of the group G are equivalent.*

(i) *G is locally supersoluble.*

(ii) (a) *The weak normalizer condition is satisfied by every finitely generated subgroup of G.*
(b) *$\{x^{\{y\}}\}$ is finitely generated for all x, y in G.*

(iii) (a) *The weak normalizer condition is satisfied by every finitely generated subgroup of G.*

(b) *If the locally noetherian subgroup X of G is normalized by the element y in G, then $\{X, y\}$ is locally noetherian.*

Remark: We have not been able to decide whether (ii. b) is indispensable or not. It is, however, clear that (ii. b) is satisfied whenever pairs of elements in G generate noetherian subgroups; and this is a consequence of (i), since locally supersoluble groups are locally noetherian. We have shown elsewhere – BAER [11; p. 62/63] – that (ii. b) is satisfied in case every element of order 0 in G is an engel element.

Proof: If G is locally supersoluble, then (ii. a) is a consequence of Lemma G.1; and it is a consequence of Theorem G.2 that G is locally noetherian, implying (ii. b).

That (ii. b) implies (iii. b), has been shown elsewhere; see BAER [11; p. 60, Lemma 4.1].

Assume finally the validity of (iii). Then we deduce from (iii. a) and Theorem G.2 that

(1) a subgroup of G is locally noetherian if, and only if, it is locally supersoluble.

Combining (1) and (iii. b) we obtain

(2) If the locally supersoluble subgroup S of G is normalized by the element s in G, then $\{S, s\}$ is locally supersoluble.

If F is a finitely generated subgroup of G, then F contains [by the Maximum Principle of Set Theory] a maximal locally supersoluble subgroup M. If x belongs to $F \cap \mathfrak{n}M$, then $\{M, x\}$ is by (2) a locally supersoluble subgroup of G; and we deduce $M = \{M, x\}$ from the maximality of M. Hence

$$M = F \cap \mathfrak{n}M.$$

Assume now by way of contradiction that $M \subset F$. We note that by (iii. a) the weak normalizer condition is satisfied by F. Because of $M = F \cap \mathfrak{n}M$ there exists an element f in F such that

$$M \subset \{M, f\} \subseteq F \text{ and } \{f^M\} \subseteq M_{\{M,f\}}\{f\}.$$

The subgroup $N = M_{\{M,f\}}\{f\}$ is normalized by M and by f and is therefore a normal subgroup of $\{M, f\}$. Since M is locally supersoluble, so is its subgroup $M_{\{M,f\}}$; and since this subgroup is normalized by f, we deduce from (2) that N is locally supersoluble. Next we note that

$$\{M, f\}/N = MN/N \cong M/(M \cap N)$$

is locally supersoluble as an epimorphic image of the locally supersoluble group M. Furthermore

$$N/M_{\{M,f\}} \cong \{f\}/[\{f\} \cap M_{\{M,f\}}]$$

is cyclic. Hence $\{M, f\}/M_{\{M,f\}}$ is an extension of its cyclic normal subgroup $N/M_{\{M,f\}}$ by the locally supersoluble and hence locally noetherian group $\{M, f\}/N$. But an extension of a cyclic group by a locally noetherian group is locally noetherian, since extensions of noetherian groups by noetherian groups are noetherian. Thus we have shown that

$$\{M, f\} = MN,\ M_{\{M,f\}} \subseteq M \cap N,$$

$$M, N \text{ and } \{M, f\}/M_{\{M,f\}} \text{ are locally noetherian.}$$

But this shows that $\{M, f\}$ too is locally noetherian; see BAER [7; p. 351, Satz 1]. Hence $\{M, f\}$ is by (1) locally supersoluble. Because of the maximality of M it follows that f belongs to $\{M, f\} = M$, a contradiction showing that $F = M$ is locally supersoluble. But F is finitely generated and hence supersoluble, proving that G is locally supersoluble. Hence conditions (i)–(iii) are equivalent.

We are now in a position to obtain an essential improvement upon Theorem G.2. The principal step of our argument is contained in the following

Lemma G.4: *The group G is locally supersoluble, if*

(a) *the weak normalizer condition is satisfied by every factor of G and if*
(b) *every epimorphic image, not* 1, *of G possesses an abelian normal subgroup, not* 1.

Proof: Suppose that the locally noetherian subgroup X of G is normalized by the element x in G. Condition (b) is inherited by the subgroup $\{X, x\} = S$ of G; see BAER [10; p. 17, Lemma 3.1]. Consider the set Σ of all the normal subgroups N of S such that $N \subseteq X$ and $\{N, x\}$ is locally noetherian. Clearly 1 is in Σ and we may apply the Maximum Principle of Set Theory. Consequently there exists a maximal subgroup M in Σ. Assume by way of contradiction that $M \subset X$. We note that M and X are normal subgroups of S. Consequently there exists a normal subgroup W of S with the following properties:

$$W' \subseteq M \subset W \subseteq X;$$

see BAER [10; p. 17, Lemma 3.2]. We note that W/M is an abelian normal subgroup of $\{W, x\}/M$ with cyclic quotient group $\{W, x\}/W$ and that the weak normalizer condition is [because of (a)] satisfied by every factor of $\{W, x\}/M$. An immediate application of Theorem C.2 – upon the finitely generated subgroups of $\{W, x\}/M$ – shows the local supersolubility of $\{W, x\}/M$; [as a consequence of the auxiliary proposition (4), verified in the course of the proof of Theorem G.2 the group $\{W, x\}/M$ is actually supersoluble] and this implies in particular that $\{W, x\}/M$ is locally noetherian. Clearly W is locally noetherian as a subgroup of X and $\{M, x\}$ is locally noetherian, since M belongs to Σ. Since furthermore

$$\{W, x\} = W\{M, x\} \text{ and } M \subseteq W \cap \{M, x\},$$

application of BAER [7; p. 351, Satz 1] shows that $\{W, x\}$ is locally noetherian. Hence W belongs to Σ, contradicting the maximality of M. Consequently $M = X$ so that $S = \{X, x\} = \{M, x\}$ is locally noetherian.

Thus we have verified the validity of condition (iii. b) of Corollary G.3; and its condition (iii. a) is a consequence of our condition (a). Hence G is locally supersoluble.

Theorem G.5: *The group G is a noetherian supersoluble group if, and only if,*

(a) *G is finitely generated,*
(b) *every epimorphic image, not 1, of G possesses an abelian normal subgroup, not 1, and*
(c) *the weak normalizer condition is satisfied by every factor of G.*

The sufficiency of these conditions is an immediate consequence of Lemma G.4 and their necessity is contained in Theorem G.2 – (a), (b), (c) is just a weakened form of Theorem G.2, (iii).

Corollary G.6: *The group G is locally supersoluble if, and only if, every finitely generated subgroup F of G meets the following two requirements:*

(a) *every epimorphic image, not 1, of F possesses an abelian normal subgroup, not 1; and*
(b) *the weak normalizer condition is satisfied by every factor of F.*

This is an immediate consequence of Theorem G.5.

Proposition G.7: *If the weak normalizer condition is satisfied by the factors of the finitely generated group G, then the following properties of G are equivalent:*

(i) *G is supersoluble.*
(ii) *G is an extension of a supersoluble group by a supersoluble group.*
(iii) *$G^{(i)}$ is supersoluble for at least one i.*
(iv) *$G^{(i)}$ is hypercentral for at least one i.*
(v) *There exists a hypercentral normal subgroup N of G such that every epimorphic image, not 1, of G/N possesses an abelian normal subgroup, not 1.*

Note: It is still an open question whether or not all finitely generated groups whose factors satisfy the weak normalizer condition are supersoluble.

Proof: It is clear that (ii) is a consequence of (i). If (ii) is true, then there exists a supersoluble normal subgroup N of G with supersoluble G/N. As a finitely generated supersoluble group G/N is noetherian [Theorem G.2]. Hence $[G/N]^{(i)} = 1$ for at least one i so that $G^{(i)} \subseteq N$; and this implies the supersolubility of $G^{(i)}$. Thus (iii) is a consequence of (ii).

If $G^{(i)}$ is supersoluble, then $G^{(i+1)}$ is hypercentral by BAER [3; p. 21, Proposition 2]. Hence (iv) is a consequence of (iii); and it is clear that (v) is a consequence of (iv).

Assume finally the existence of a hypercentral normal subgroup N of G such that every epimorphic image, not 1, of G/N possesses an abelian normal subgroup, not 1. Consider an epimorphism σ of G upon $H \neq 1$. If firstly $N^\sigma = 1$, then H is an epimorphic image of G/N and possesses consequently an abelian normal subgroup, not 1. If secondly $N^\sigma \neq 1$, then we deduce $\mathfrak{z}N^\sigma \neq 1$ from the hypercentrality of N. Thus $\mathfrak{z}N^\sigma$ is an abelian normal subgroup, not 1, of H. Consequently we have shown that G meets requirements (a), (b), (c) of Theorem G.5, proving the supersolubility of G and the equivalence of (i)–(v).

H. Theorem: *The group G is supersoluble if, and only if,*
(a) *the weak normalizer condition is satisfied by every factor of G and*
(b) *every locally supersoluble subgroup of G is supersoluble.*

Proof: Every factor of a supersoluble group is supersoluble. Hence (a) is a consequence of Lemma G.1 and the necessity of (b) is obvious.

Assume conversely the validity of conditions (a) and (b). Every group possesses a maximal locally supersoluble subgroup. Hence there exists a maximal locally supersoluble subgroup M of G. It is a consequence of (b) that M is a maximal supersoluble subgroup of G.

Assume by way of contradiction that $M \subset \mathfrak{n}M$. Then there exists an element c in $\mathfrak{n}M$ which does not belong to M. Let $C = M\{c\}$. Then M is a supersoluble normal subgroup of C with cyclic C/M. If F is a finitely generated subgroup of C, then $F \cap M$ is a supersoluble normal subgroup of F with cyclic

$$F/(F \cap M) \cong MF/M \subseteq C/M.$$

Since the weak normalizer condition is satisfied by the factors of G and hence of the finitely generated subgroup F of G, application of Proposition G.7 shows the supersolubility of F. Hence C is locally supersoluble. But $M \subset C$, contradicting the maximality of M. Hence

$$M = \mathfrak{n}M.$$

Assume by way of contradiction that $M \subset G$. Application of the weak normalizer condition shows the existence of an element t in G, not in M, such that

$$\{t^M\} \subseteq M_{\{M,t\}}\{t\}.$$

If σ is the canonical epimorphism of $\{M, t\} = T$ upon $L = \{M, t\}/M_{\{M,t\}}$, then $\{t^\sigma\} = \{(t^\sigma)^L\}$ is a cyclic normal subgroup of L and

$$L/\{t^\sigma\} = M^\sigma\{t^\sigma\}/\{t^\sigma\} \cong M^\sigma/[M^\sigma \cap \{t^\sigma\}]$$

is an epimorphic image of the supersoluble group M and hence supersoluble. But an extension of a cyclic normal subgroup by a supersoluble group is [trivially] supersoluble. Hence L is supersoluble.

If E is a finitely generated subgroup of $\{M, t\}$, then

$$E/[E \cap M_{\{M,t\}}] \cong M_{\{M,t\}}E/M_{\{M,t\}} \subseteq L$$

is supersoluble and $E \cap M_{\{M,t\}} \subseteq M$ is likewise supersoluble. Thus E is a finitely generated extension of a supersoluble group by a supersoluble group and the weak normalizer condition is satisfied by the factors of E, since it is satisfied by the factors of G. Application of Proposition G.7 shows the supersolubility of E. Hence $\{M, t\}$ is locally supersoluble. This contradicts the maximality of M, since $M \subset \{M, t\}$. Hence $G = M$ is supersoluble.

Corollary: *The group G is noetherian and supersoluble if, and only if,*

(a) *G is finitely generated,*

(b) *the weak normalizer condition is satisfied by every factor of G and*

(c) *every locally supersoluble subgroup of G is supersoluble.*

This is an almost immediate consequence of our preceding Theorem.

I. Lemma: *Every supersoluble group G has the property:*

(+) *If S is a subgroup of the factor F of G with $\mathfrak{n}S = S \subset F$, and if T is a normal subgroup of F with $F = ST$, then there exists an element t in T with*

$$S \subset \{S, t\} \text{ and } \{t^S\} \subseteq S_{\{S,t\}}\{t\}.$$

Proof: If F is a factor of the supersoluble group G, then F too is supersoluble. Consider now a subgroup S with $S \subset F$ and a normal subgroup T of F with $F = ST$. Then

$$S/S_F \subset F/S_F = [S/S_F][S_FT/S_F]$$

so that in particular the normal subgroup S_FT/S_F of F/S_F is not 1. Consequently there exists a normal subgroup V of F with

$$S_F \subset V \subseteq S_FT \text{ and cyclic } V/S_F.$$

This implies the existence of an element t in T with $V/S_F = \{S_Ft\}$. Since $V = \{S_F, t\}$, and since V as a normal subgroup of F which is greater than S_F cannot be part of S, it follows that t does not belong to S. Since V is normal in F, we find that

$$\{t^S\} \subseteq \{t^F\} \subseteq V = S_F\{t\} \subseteq S_{\{S,t\}}\{t\},$$

as S_F is a normal subgroup of F and $\{S, t\} \subseteq F$.

Note: The property (+) has been derived for all subgroups $S \subset F$ and all normal subgroups T with $F = ST$ without the restriction to normalizer-equal subgroups S. But we shall use the property $(+)$ exactly in the form stated in the Lemma and we shall term this property $(+)$ *the cross condition.* If we apply, as is possible, $(+)$ with the special choice $F=T$, then we see that $(+)$ implies the weak normalizer condition for all factors of G. The cross condition is consequently a strong form of the weak normalizer condition.

Theorem: *The cross condition implies local supersolubility.*

Proof: Suppose that the group G satisfies the cross condition. Then as has been pointed out in the preceding Note

(1) The weak normalizer condition is satisfied by all the factors of G.

Consider a locally noetherian subgroup S of G and an element t in G normalizing S. Assume by way of contradiction that $\{S, t\}=S\{t\}=T$ is not locally noetherian. There exists a maximal locally noetherian subgroup M of T which contains t [Maximum Principle of Set Theory]. Clearly S is a normal subgroup of T and $T=SM$. If the element a in T normalizes M, then $a=mb$ with m in M and b in S; and M is also normalized by b.

(a) $\{M, a\}=\{M, b\}=M\{M \cap S, b\}$.

(b) M and $\{M \cap S, b\}$ are locally noetherian, as follows from the choice of M and from $\{M \cap S, b\} \subseteq S$,

(c) M is a normal subgroup of $\{M, a\}$, since M is normalized by a.

Since the normal subgroup S of T is normalized by $\{M, a\}$ it follows that $M \cap S$ is a normal subgroup of $\{M, a\}$. Since M is normalized by the element b in S, we find that $M \circ b$ is part of $M \cap S$; and this implies that b^M is contained in $\{M \cap S, b\}$. Thus $\{M \cap S, b\}$ is normalized by M and b; and we have shown that

(d) $\{M \cap S, b\}$ is a normal subgroup of $\{M, b\}=\{M, a\}$.

The facts (a)–(d) show that $\{M, a\}$ is the product of two locally noetherian normal subgroups. Hence $\{M, a\}$ is locally noetherian by Baer [7; p. 353, Folgerung 1]. Because of the maximality of M we have therefore $M=\{M, a\}$, proving $M=T \cap \mathfrak{n}M \subset T$, as M is and T is not locally noetherian. Recall that S is a normal subgroup of T and that $T=SM$. Application of the cross condition shows the existence of an element s in S with

$$M \subset \{M, s\} \text{ and } \{s^M\} \subseteq M_{\{M,s\}}\{s\}.$$

Now $\{s^M\}$ is a normal subgroup of $\{M, s\}$; and $\{s^M\} \subseteq S$, since S is a normal subgroup of $T=MS$. As S is locally noetherian, so is its subgroup $\{s^M\}$. Naturally $M_{\{M,s\}}$ is, as a subgroup of M, a locally noetherian normal subgroup of $\{M, s\}$. Thus $M_{\{M,s\}}\{s\}=M_{\{M,s\}}\{s^M\}$ is the product of two locally noetherian normal subgroups of $\{M, s\}$; and as such it is a locally noetherian normal subgroup of $\{M, s\}$; see BAER [7; p. 353, Folgerung 1]. Hence

(A) $\{M, s\} = M\ [M_{\{M,s\}}\{s\}]$ and $M_{\{M,s\}} \subseteq M \cap M_{\{M,s\}}\{s\}$;

(B) M and $M_{\{M,s\}}\{s\}$ are locally noetherian;

(C) $M_{\{M,s\}}\{s\}$ is a normal subgroup of $\{M, s\}$.

Clearly $[M_{\{M,s\}}\{s\}]/M_{\{M,s\}}$ is cyclic and

$$\{M, s\}/M_{\{M,s\}}\{s\} \cong M/[M \cap M_{\{M,s\}}\{s\}]$$

is locally noetherian as an epimorphic image of the locally noetherian group M. It is easily verified that extensions of cyclic groups by locally noetherian groups are locally noetherian, since extensions of noetherian groups by noetherian groups are noetherian. Hence

(D) $\{M, s\}/M_{\{M,s\}}$ is locally noetherian.

Because of (A)–(D) we may apply BAER [7; p. 351, Satz 1] to show that $\{M, s\}$ is locally noetherian. Because of the maximality of M it follows that s belongs to $\{M, s\} = M$, contradicting our choice of s. This contradiction shows that T is locally noetherian. Hence we have shown:
(2) If the locally noetherian subgroup S of G is normalized by the element s in G, then $\{S, s\}$ is locally noetherian.

Because of (1) and (2) we may apply Corollary G.3 to show the local supersolubility of G.

Corollary 1: *The group G is noetherian and supersoluble if, and only if, G is finitely generated and the cross condition is satisfied by G.*

This is an almost immediate consequence of this section's Lemma and Theorem.

Corollary 2: *The group G is locally supersoluble if, and only if, the cross condition is satisfied by the finitely generated subgroups of G.*

An immediate consequence of Corollary 1.

J. It is known that the product of two finite supersoluble normal subgroups need not be supersoluble; see BAER [8; p. 186, Example 1]. But such a finite product will be supersoluble if, and only if, the commutator subgroup of the product is nilpotent; see BAER [8; p. 186, Corollary 2]. We are going to generalize this result.

Lemma J.1: *Every product of a supersoluble and a hypercentral normal subgroup is supersoluble.*

Proof: If $H \neq 1$ is an epimorphic image of such a group G, then there exist normal subgroups A and B of H with $H = AB$ and A hypercentral and B supersoluble. If $A = 1$, then $H = B$ is supersoluble and possesses therefore a cyclic normal subgroup, not 1. Assume next that

$A \neq 1$. Then $\mathfrak{z}A \neq 1$, since A is hypercentral; and $\mathfrak{z}A$ is a normal subgroup of H as a characteristic subgroup of a normal subgroup. If $B \cap \mathfrak{z}A = 1$, then $\mathfrak{z}A$ is centralized by B and A and hence by $H = AB$ so that $1 \subset \mathfrak{z}A \subseteq \mathfrak{z}H$. Clearly there exists a cyclic normal subgroup, not 1, of H in this case. Assume finally that $K = B \cap \mathfrak{z}A \neq 1$. Then K is a normal subgroup, not 1, of the supersoluble group B. Consequently there exists a cyclic normal subgroup Z of B with $1 \subset Z \subseteq K$; see BAER [3; p. 17, Lemma 2]. Since Z is centralized by A and normalized by B, it is normalized by $AB = H$. Thus we have shown in every case the existence of a cyclic normal subgroup, not 1, of H, proving the supersolubility of G.

Lemma J.2: *If $G = AB$ is the product of its supersoluble normal subgroups A and B, then the following properties of G are equivalent.*

(i) *G is supersoluble.*
(ii) *G' is hypercentral.*
(iii) *$A \circ B$ is hypercentral.*

Proof: That (ii) is a consequence of (i), is contained in BAER [3; p. 21, Proposition 2]. It is furthermore clear that (iii) is a consequence of (ii). That (ii) is a consequence of (iii), may be inferred from $G' = A'(A \circ B)B'$, the hypercentrality of the commutator subgroup of supersoluble groups, just referred to, and Proposition D.2.

Assume finally the validity of (ii) and consider an epimorphic image $H \neq 1$ of G. Then $H = JK$ is the product of its supersoluble normal subgroups J and K and H' is hypercentral. Application of Lemma J.1 shows that $L = JH'$ and $M = KH'$ are supersoluble normal subgroups of H with $H = LM$ and $H' \subseteq L \cap M$.

If $H' = 1$, then H is abelian and possesses certainly cyclic normal subgroups, not 1. Thus we assume in the sequel that $H' \neq 1$. This implies

$$1 \subset \mathfrak{z}H' \subseteq H' \subseteq L \cap M,$$

since H' is hypercentral. We note furthermore that $\mathfrak{z}H'$ is a characteristic subgroup of H and hence a normal subgroup of L and M.

Since $\mathfrak{z}H'$ is a normal subgroup, not 1, of the supersoluble group L, there exists a cyclic normal subgroup Z of L with $1 \subset Z \subseteq \mathfrak{z}H'$; see BAER [3; p. 17, Lemma 2]. Then

$$N = \{Z^H\} = \{Z^{LM}\} = \{Z^M\} \subseteq \mathfrak{z}H' \subseteq L \cap M$$

so that N is an abelian normal subgroup of H. Since N is centralized by H', the group Σ of automorphisms, induced in N by H, is abelian. If we denote by M^* the subgroup of those automorphisms in Σ which are induced in N by elements in M, then $M^* \subseteq \Sigma$ and

$$N = \prod_{\sigma \varepsilon M^*} Z^\sigma,$$

since N is abelian. Let λ be an automorphism of N which is induced in

N by an element in L. Since Z is a normal subgroup of L, we have $Z=Z^\lambda$; and since Z is cyclic, there exists an integer $k\neq 0$ such that $x^\lambda=x^k$ for every x in Z. If σ is an automorphism in M^* and y an element in Z^σ, then there exists an element x in Z with $y=x^\sigma$. Since σ and λ belong to the commutative group Σ of automorphisms of N, it follows that

$$y^\lambda=x^{\sigma\lambda}=x^{\lambda\sigma}=x^{k\sigma}=x^{\sigma k}=y^k.$$

Since the abelian group N is the product of all the Z^σ for σ in M^*, we conclude that

$$s^\lambda=s^k \text{ for every } s \text{ in } N.$$

It follows that every subgroup of N is mapped upon itself by the automorphism λ. Thus we have shown that every element in L induces an automorphism of N which maps every subgroup of N onto itself; and this is equivalent to saying that

(+) every subgroup of N is normalized by L.

Since $N\neq 1$ is a normal subgroup of the supersoluble group M, there exists a cyclic normal subgroup C of M with $1\subset C\subseteq N$; see Baer [3; p. 17, Lemma 2]. But C is normalized by L too – see (+) – so that C is normalized by $ML=H$. Thus we have shown in either case the existence of a cyclic normal subgroup, not 1, of H, proving the supersolubility of G and the equivalence of (i)–(iii).

Proposition J.3: *If G is the product of finitely many supersoluble normal subgroups $N_1, \dots, N_n$, then the following properties are equivalent:*

(i) *G is supersoluble.*
(ii) *G' is hypercentral.*
(iii) *$N_i \circ N_j$ is hypercentral for $i\neq j$.*

The simple derivation of this result from Lemma J.2 may be left to the reader.

Theorem J.4: *If the group G of finite abelian subgroup rank is the product of its supersoluble normal subgroups, then the following properties of G are equivalent.*

(i) *G is supersoluble.*
(ii) *G' is hypercentral.*
(iii) *$X \circ Y$ is hypercentral for supersoluble normal subgroups X, Y of G.*

Proof: That (i) implies (ii), is a consequence of Baer [3; p. 21, Proposition 2]; and it is obvious that (ii) implies (iii). Using Proposition J.3 we deduce from (iii) the following property:

(iv) Products of finitely many supersoluble normal subgroups of G are supersoluble.

If F is a finite subset of the product G of its supersoluble normal subgroups, then F is contained in a product of finitely many supersoluble normal subgroups. This product is, by (iv), supersoluble; and thus $\{F\}$ too is supersoluble. Hence G is locally supersoluble. But G is of finite abelian rank and the supersolubility of G is a consequence of Theorem F.2, showing the equivalence of (i)–(iv).

LITERATURE

BAER, REINHOLD, [1] The hypercenter of a group. Acta Math. **89**, 165–208 (1953).

————, [2] Das Hyperzentrum einer Gruppe III. Math. Zeitschr. **59**, 299–338 (1953).

————, [3] Supersoluble groups. Proc. Amer. Math. Soc. **6**, 16–32 (1955).

————, [4] Auflösbare Gruppen mit Maximalbedingung. Math. Ann. **129**, 139–173 (1955).

————, [5] Finite extensions of abelian groups with minimum condition. Trans. Amer. Math. Soc. **79**, 521–540 (1955).

————, [6] Noethersche Gruppen. Math. Zeitschr. **66**, 269–288 (1956).

————, [7] Lokal Noethersche Gruppen. Math. Zeitschr. **66**, 341–363 (1957).

————, [8] Classes of finite groups and their properties. Illinois Journal Math. **1**, 115–187 (1957).

————, [9] Überauflösbare Gruppen. Abh. Math. Seminar Hamburg **23**, 11–28 (1959).

————, [10] Gruppen mit Minimalbedingung. Math. Ann. **150**, 1–44 (1963).

————, [11] Erreichbare und engelsche Gruppenelemente. Abh. Math. Seminar Hamburg **27**, 44–74 (1964).

————, [12] Nilgruppen. Math. Zeitschr. **62**, 402–439 (1955).

FUCHS, L., Abelian groups, Budapest 1958.

HALL, M. JR., The Theory of Groups. New York 1959.

HIRSCH, KURT, Über lokal nilpotente Gruppen. Math. Zeitschr. **63**, 290–294 (1955).

KUROSH, A. G., The theory of groups I, II. Second english edition, New York 1960.

MAL'CEV, A. I., On some classes of finite soluble groups. Mat. Sbornik **28**, 567–588 (1951).

PLOTKIN, B. I., The nilradical of a group. Dokl. Akad. Nauk USSR **98**, 341–343 (1954).

ZASSENHAUS, HANS, Lehrbuch der Gruppentheorie I. Leipzig–Berlin 1937.